

# Prüfer's Ideal Numbers as Gelfand's maximal Ideals

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**ABSTRACT.** Polyadic arithmetics is a branch of mathematics related to  $p$ -adic theory. The aim of the present paper is to show that there are very close relations between polyadic arithmetics and the classic theory of commutative Banach algebras. Namely, let  $\mathcal{A}$  be the algebra consisting of all complex periodic functions on  $\mathbb{Z}$  with the uniform norm. Then the polyadic topological ring can be defined as the ring of all characters  $\mathcal{A} \rightarrow \mathbb{C}$  with convolution operations and the Gelfand topology.

## Prolegomena

Let us start from an elementary polynomial identity. Let  $\{p_j\}$  be an arbitrary sequence of natural numbers and  $z$  be a complex variable. It is easily seen that for each  $k \in \mathbb{N}$

$$(1+z+\dots+z^{p_1-1})(1+z^{p_1}+\dots+z^{p_1(p_2-1)})(1+z^{p_1p_2}+\dots+z^{p_1p_2(p_3-1)})\dots = \\ = 1 + z + z^2 + z^3 + \dots + z^{p_1p_2\cdots p_k-1}.$$

In particular, if the sequence  $p_j$  is constant, i.e.  $p_j = p$  for all  $j$ , the identity will be transformed to the form

$$\prod_{r=0}^{k-1} \left( \sum_{\nu=0}^{p-1} z^{\nu p^r} \right) = \sum_{n=0}^{p^k-1} z^n$$

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showing that each non-negative integer  $n < p^k$  can be uniquely represented by the sum

$$(1) \quad n = \nu_0(n) \cdot p^0 + \nu_1(n) \cdot p^1 + \cdots + \nu_{k-1}(n) \cdot p^{k-1},$$

where *the digits*  $\nu_r(n)$  satisfy the following condition:  $0 \leq \nu_r(n) < p$ . If we set  $p_j = j$  then  $p_1 \cdots p_r = r!$  and the identity has the form

$$\prod_{r=0}^{k-1} \frac{1 - z^{(r+1)!}}{1 - z^{r!}} = \prod_{r=0}^{k-1} \left( \sum_{\nu=0}^r z^{\nu r!} \right) = \sum_{n=0}^{k!-1} z^n$$

which shows that each non-negative integer  $n < k!$  admits the unique *factorial representation*

$$(2) \quad n = \nu_1(n) \cdot 1! + \nu_2(n) \cdot 2! + \cdots + \nu_{k-1}(n) \cdot (k-1)!$$

where *the digits*  $\nu_r(n)$  satisfy the condition  $0 \leq \nu_r(n) \leq r$ .

Trying to give a general meaning to formulae of the type (1) and (2) with an *infinite* number of digits caused H. Prüfer [2] to create a new theory of *ideal numbers*. J. von Neumann [3] further developed this theory which was later completed in the theory of adeles ([9]).

The representation (1) of ideal numbers with the prime  $p$  as the radix and an infinite number of digits (invented by K. Hensel [1]) gave rise to the *p-adic analysis* (for references see e.g. [19]).

The factorial representation (2) with an infinite number of digits lays at the foundation of the *polyadic analysis*. M. D. Van Dantzig [4] and later E. Novoselov [8] gave an essential contribution to this area. The most significant results were summarized by E. Novoselov in his book [13].

By a polyadic number E. Novoselov means a *finite integral adele*. Such a number can be viewed as a limit of an integer sequence which converges *p*-adically for every prime  $p$ . The Novoselov's theory includes the polyadic topology, arithmetics, the basic analytic functions, measure theory and integration in the polyadic domain.

J.-L. Mauclore well adapted and completed this theory in a number of papers [12, 14, 16, 17].

The most important property of the polyadic topological ring  $\mathcal{P}$  with the divisibility topology  $\sigma$  is to be a *compact Hausdorff space*. As is well known, there is no translation-invariant  $\sigma$ -additive measure on  $\mathbb{Z}$ . But such a measure (Haar measure) exists on the compact additive group of the polyadic ring  $(\mathcal{P}, \sigma)$  including  $\mathbb{Z}$  as dense subset. So, the *compactification* of the ring  $\mathbb{Z}$  is the key idea of the Novoselov theory. His method for the investigation of the distribution problems of arithmetic functions has been studied by several papers and books, such as [10, 15, 18, 20, 21].

Of course, various ways of compactification are possible. They lead us to different (a priori) completions of  $\mathbb{Z}$ .

Z. Krizhyus [15] deals with the compactification based on *periodicity*, which seems very natural. Indeed, on the one hand, the sum (or the product) of  $m$ - and  $n$ -periodic functions on  $\mathbb{Z}$  has each common multiple of  $m$  and  $n$  as period. On the other hand, if a function  $f$  is  $m$ - and  $n$ -periodic simultaneously then any common divisor of  $m$  and  $n$  is also a period of  $f$ . As divisibility is a main notion of arithmetics, the algebra  $\mathcal{A}$  of all periodic function on  $\mathbb{Z}$  must play an important role in number theory.

In fact, there are many known works (cf. [6, 7, 10, 11, 18, 20, 21]) concerned with the *almost periodic* functions which are elements of closures  $\mathcal{A}$  with respect to various norms. In [15] the *uniform* completion  $\mathcal{A}$  of the algebra  $\mathcal{A}$  is considered. By the theory of commutative Banach algebras presented, e. g., in [5],  $\mathcal{A}$  is isometrically isomorphic to the algebra of all continuous functions on the *Gelfand  $\mathcal{A}$ -compactification*  $(\mathcal{G}, \gamma)$  of  $\mathbb{Z}$ .

The plan of this article is as follows. In the sections 1–3 we introduce the basic concepts and notations used in the paper. Then, in the section 4 we define the ring operations in  $\mathcal{G}$ . Further, in the section 5 we define a *cluster topology* in the ring  $\mathcal{G}$  and we prove that it is identical with the Gelfand topology  $\gamma$ . Next, in the sections 6–7 the polyadic topological ring  $(\mathcal{P}, \sigma)$  is described. This exposition differs from Novoselov's one merely in non-essential technical details. The contents of the section 8 can be regarded as a presentation of some “bridge” between the topological rings  $(\mathcal{P}, \sigma)$  and  $(\mathcal{G}, \gamma)$ . At last, in the section 9 we state the identity of these topological rings.

## 1. Preliminaries

**1.1. Weak topologies in the dual space.** Let  $B_0$  be a dense subspace of a Banach space  $B$ . For any  $\psi \in B^*$ ,  $u \in B_0$  and  $R > 0$  we define a *neighborhood*  $U_R(u; \psi)$  of the element  $\psi$ :

$$U_R(u; \psi) = \{\phi \in B^* : |\phi(u) - \psi(u)| < R\}.$$

These neighborhoods form a subbase of some topology in  $B^*$  called  $B_0$ -topology. If  $B_0 = B$  then it is called the *\*-weak topology* in  $B^*$ .

**PROPOSITION 1.** *All topologies induced by different  $B_0$ -topologies on a bounded subset  $\mathcal{F} \subseteq B^*$  coincide.*

**PROOF.** Let us check that a topology  $\tau$  induced on  $\mathcal{F}$  by an arbitrary  $B_0$ -topology coincides with the topology  $\sigma$  induced on this set by the \*-weak topology of the space  $B^*$ . Since it is obvious (from the

definition of the  $B_0$ -topology) that  $\tau \subseteq \sigma$ , it is necessary to show that  $\sigma \subseteq \tau$ . It will suffice to prove that for each functional  $\psi \in \mathcal{F}$  an arbitrary element  $U_R(u; \psi)$  of the  $\sigma$ -subbase is wider than some element  $U_r(v; \psi)$  from the  $\tau$ -subbase. Let the norms of functionals from  $\mathcal{F}$  be bounded by  $N$ . Let us choose an element  $v$  of the set  $B_0$  (dense in  $B$ ) and a positive number  $r$  such that the following inequalities

$$3\|u - v\| < R/N, \quad 3r < R$$

hold. Under this condition  $U_r(v; \psi) \subseteq U_R(u; \psi)$ . Indeed, for any functional  $\phi$  from  $U_r(v; \psi)$  we have

$$\begin{aligned} |\phi(u) - \psi(u)| &\leq |\phi(u - v)| + |\phi(v) - \psi(v)| + |\psi(v - u)| \leq \\ &\leq \|\phi\| \cdot \|u - v\| + |\phi(v) - \psi(v)| + \|\psi\| \cdot \|u - v\| < \\ &< N \cdot \|u - v\| + r + N \cdot \|u - v\| < R/3 + R/3 + R/3, \end{aligned}$$

i. e. every functional  $\phi$  from  $U_r(v; \psi)$  belongs to  $U_R(u; \psi)$ .  $\square$

**1.2. Topological ring.** A subset  $U$  of a commutative ring  $R$  is called *symmetric* if with each element  $a \in U$  it contains its opposite  $(-a)$ . In order to define in  $R$  a Hausdorff topology compatible with the algebraic structure of  $R$ , it is convenient to set a *base of neighborhoods of zero* that is a class  $\mathcal{B}$  of symmetric subsets complying with the following conditions:

- (1) The zero of the ring  $R$  is a unique common element of all neighborhoods.
- (2) An intersection of two arbitrary neighborhoods of zero includes some neighborhood of zero.
- (3) For each neighborhood of zero  $V$  it is possible to find a neighborhood of zero  $U$  such that  $\{u_1 + u_2 : u_1, u_2 \in U\} \subseteq V$ .
- (4) For each neighborhood of zero  $V$  and arbitrary element  $a \in R$  it is possible to show a neighborhood  $U$  such that  $\{au : u \in U\} \subseteq V$ .
- (5) For an arbitrary neighborhood of zero  $V$  it is possible to specify a neighborhood of zero  $U$  such that  $\{u_1 u_2 : u_1, u_2 \in U\} \subseteq V$ .

The *translations*  $U_a = \{a + u : u \in U\}$  of neighborhoods of zero form a base  $\mathcal{B}_a$  of neighborhoods of an arbitrary point  $a$ :

$$\mathcal{B}_a = \{U_a : U \in \mathcal{B}\}.$$

An empty set and any subset  $S \subseteq R$  including some neighborhood  $U_a$  of each its point  $a$  are announced as *open* in the topology of the ring  $R$ . Obviously, by this definition, a union of open sets and an intersection of two open sets are open.

DEFINITION 1. *The interior  $W_a^o$  of a neighborhood  $W_a$  consists of points  $U$  having a neighborhood of zero  $V$  such that  $V_u \subseteq W_a$ .*

The interior  $W_a^o$  is not empty, because it contains the point  $a$ .

PROPOSITION 2. *The set  $W_a^o$  is the largest open part of  $W_a$ .*

PROOF. Let  $u \in W_a^o$  and let  $V$  be a neighborhood of zero for which  $V_u \subseteq W_a$ . Let  $U$  be a neighborhood of zero found according to (3). Let us check that  $U_u \subseteq W_a^o$  and thus establish that  $W_a^o$  is open.

The neighborhood  $U_v$  of an arbitrary point  $v \in U_u$  consists of points  $u + u_1 + u_2$  where  $u_1, u_2$  belong to the neighborhood  $U$ . By virtue of (3),  $u_1 + u_2 \in V$ , so that  $U_v \subseteq V_u \subseteq W_a$ . Hence, the arbitrary point  $v$  of the neighborhood  $U_u$  belongs to  $W_a^o$ , as was to be checked.

Let now  $S$  be a nonempty open subset of a neighborhood  $W_a$ . Each point  $u$  belongs to  $S$  together with some neighborhood  $V_u$ . All the more,  $u$  belongs to  $W_a$  together with  $V_u$ , i. e.  $u \in W_a^o$ .  $\square$

PROPOSITION 3. *Let topologies  $\sigma$  and  $\tau$  in  $R$  be defined by two bases of neighborhoods  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. The topology  $\sigma$  is weaker than the topology  $\tau$  if and only if each neighborhood  $U \in \mathcal{A}$  includes some neighborhood  $V \in \mathcal{B}$ .*

PROOF. Let  $\sigma$  be weaker than  $\tau$ , i. e. every  $\sigma$ -open set is  $\tau$ -open. The interior  $U^o$  of the neighborhood  $U$  contains the zero of the ring  $R$  and it is  $\sigma$ -open. A fortiori it is  $\tau$ -open, so that zero is included into it with some  $\tau$ -neighborhood  $V$ . Thus,  $V \subseteq U^o \subseteq U$ .

Conversely, each point  $s$  of the  $\sigma$ -open set  $S$  belongs to it with a neighborhood  $U_s$  which is a translation of a neighborhood of zero  $U$ . The neighborhood  $U$  contains some neighborhood  $V \in \mathcal{B}$ . Since the above translation keeps the inclusion of sets  $V_s \subseteq U_s \subseteq S$ , the point  $s$  of  $S$  belongs to  $S$  with a  $\tau$ -neighborhood  $V_s$ . Hence this set is  $\tau$ -open.  $\square$

## 2. Almost periodic functions on $\mathbb{Z}$

**2.1. Algebra  $\mathcal{A}$  of almost periodic functions.** Denote by  $\mathcal{A}$  the class of all complex-valued periodic functions  $u$  on  $\mathbb{Z}$ . This is a commutative algebra with the unit  $e$ , termwise operations and a complex conjugation  $u \mapsto u^*$  as involution. The class  $\mathcal{A}_p$  consisting of all  $p$ -periodic complex-valued functions on  $\mathbb{Z}$  is a finite-dimensional subalgebra in  $\mathcal{A}$ . Let  $p$  and  $q$  be arbitrary positive integers. Any  $q$ -periodic function is also a  $pq$ -periodic one. Hence  $\mathcal{A}$  can be considered as an *inductive limit* of the increasing sequence of algebras

$$\mathcal{A}_{1!} \subseteq \mathcal{A}_{2!} \subseteq \cdots \subseteq \mathcal{A}_{n!} \subseteq \cdots.$$

A periodic function  $u$  on  $\mathbb{Z}$  takes only a finite number of values, hence one can define on  $\mathcal{A}$  a uniform norm  $\|u\| = \sup |u(n)|$  having the following properties:  $\|e\| = 1$ ;  $\|uv\| \leq \|u\| \cdot \|v\|$ ;  $\|u^*u\| = \|u\|^2$ . By completion of  $\mathcal{A}$  we get the Banach algebra  $\mathcal{A}$  of *almost periodic* functions on  $\mathbb{Z}$ .

## 2.2. Subalgebra of $p$ -periodic functions.

DEFINITION 2. Denote by  $\text{Res}_p(n)$  a *residuum* under division  $n \in \mathbb{Z}$  by  $p \in \mathbb{N}$  defined by the equality  $n = mp + \text{Res}_p(n)$  ( $0 \leq n - mp < p$ ). We define a *raster*  $R_p(m)$  as the set of level  $m$  of the function  $\text{Res}_p$ :

$$R_p(m) = \{k \in \mathbb{Z} : \text{Res}_p(k) = m\} \quad (0 \leq m < p)$$

PROPOSITION 4. The indicators  $e_m^p$  ( $0 \leq m < p$ ) of the rasters  $R_p(m)$  have the following properties:

- (1)  $e_m^p e_n^p = \delta_{mn} e_m^p$ ;
- (2)  $\sum e_m^p = e$ ;
- (3)  $u = \sum u(m) e_m^p$  for any  $u$  from  $\mathcal{A}_p$ ;
- (4) the functions  $e_m^p$  form a base in  $\mathcal{A}_p$ .

PROOF. (1) The product of indicators is the indicator of the intersection, but the sets of different levels do not intersect, therefore if  $m \neq n$  then  $e_m^p e_n^p = 0$ . Moreover by definition any indicator coincides with its square.

(2) The rasters  $R_p(m)$  (as sets of all different levels) split  $\mathbb{Z}$  into components. But in case of partition, the indicator  $e$  of a union is equal to the sum of the component indicators  $e_m^p$ .

(3) On the raster  $R_p(m)$  the product  $(u - u(m)e) e_m^p$  is equal to zero because of the first factor, on its completion it is equal to zero because of the second one. Therefore  $ue_m^p = u(m)e_m^p$  and hence, in view of equality (2),

$$u = ue = u \sum e_m^p = \sum ue_m^p = \sum u(m)e_m^p.$$

(4) The condition  $e_m^p(k) = 1$  is equivalent to

$$\text{Res}_p(k) = \text{Res}_p(k + p) = m.$$

The latter equality means that  $(k + p) \in R_p(m)$ . Thus,  $e_m^p(k + p) = 1$ , and the  $p$ -periodicity of the indicator  $e_m^p$  is established. Let us check the independence of the elements  $e_m^p$  whose completeness was proved in the previous item. Multiplying both parts of the equality  $\sum \alpha_m e_m^p = 0$

by  $e_n^p$  we get, in view of the item (1), the equality  $\alpha_n e_n^p = 0$  which is possible only if  $\alpha_n = 0$ .  $\square$

PROPOSITION 5. *The subalgebra  $\mathcal{A}_p$  separates the points of the set*

$$K = \{0, 1, \dots, p-1\}.$$

PROOF. Let  $0 \leq m < p$ , then  $e_m^p(k) = 1$  for  $k = m$ , therefore  $e_m^p(k) = 0$  for all other values  $k \in K$ .  $\square$

### 3. Characters of the Banach algebra $\mathcal{A}$

**3.1. General information.** We call by *character* any non-zero multiplicative linear functional  $\psi : \mathcal{A} \rightarrow \mathbb{C}$ . Such a functional is always bounded and its norm is equal to 1. Denote the set of all characters by  $\mathcal{G}$ . A non-trivial ideal of the algebra  $\mathcal{A}$  is called *maximal* if it is not a subset of any other non-trivial ideal. The kernel of a character is a maximal ideal and conversely: any maximal ideal is the kernel of a unique character.

For each element  $u \in \mathcal{A}$  we define a function  $\hat{u} : \mathcal{G} \rightarrow \mathbb{C}$  by the formula  $\hat{u}(\psi) = \psi(u)$ . The topology  $\gamma$  induced on  $\mathcal{G}$  by the  $*$ -weak topology of the dual  $\mathcal{A}^*$  is called *Gelfand topology*. It is the weakest topology for which all functions  $\hat{u}$  are continuous. Since the subalgebra  $\mathcal{A}$  is dense in  $\mathcal{A}$  and  $\mathcal{G}$  is a subset of the unit sphere by virtue of Proposition 1, it is possible to define the topology  $\gamma$  by such a subbase of neighborhoods of points  $\psi_0 \in \mathcal{G}$ :

$$(3) \quad U_R(u; \psi_0) = \{\psi \in \mathcal{G} : |\psi(u) - \psi_0(u)| < R\} \quad (u \in \mathcal{A}, R > 0).$$

Owing to classical inferences of the general Gelfand theory, the topological space  $(\mathcal{G}, \gamma)$  is a compact Hausdorff one and the transform  $G : u \mapsto \hat{u}$  is an isometry of the algebra  $\mathcal{A}$  onto the algebra of all continuous complex-valued functions  $C(\mathcal{G}, \gamma)$  with a uniform norm.

In view of the density of  $\mathcal{A}$  in the Banach algebra  $\mathcal{A}$  both algebras have the same dual and the same set of characters. Thus it is sufficient to describe only the characters of the algebra  $\mathcal{A}$ .

**3.2. Restriction of character on the subalgebra  $\mathcal{A}_p$ .** For all  $k \in \mathbb{N}$  we define a functional  $\delta_k$  by the equality  $\delta_k(u) = u(k)$ . Its linearity and multiplicativity are obvious, while its non-triviality follows from the equality  $\omega_k(e) = e(k) = 1$ .

Denote by  $\psi_p$  the restriction  $\psi|_{\mathcal{A}_p}$  of a character  $\psi$  on the subalgebra  $\mathcal{A}_p$ . As the following proposition shows, the set of non-zero restrictions  $\psi_p$  consists only of the functionals  $\delta_k$ .

**PROPOSITION 6.** *For each  $p \in \mathbb{N}$  and  $\psi \in \mathcal{G}$  either the restriction  $\psi_p$  is trivial or there is a unique non-negative number  $k = \kappa(p; \psi) < p$  such that  $\psi_p = \delta_k$ .*

**PROOF.** For a character  $\psi$  with a non-zero restriction  $\psi_p$  there is an element  $u$  in the subalgebra  $\mathcal{A}_p$  such that  $\psi(u) \neq 0$ . Dividing by  $\psi(u)$  both members of the equality  $\psi(u) = \psi(eu) = \psi(e)\psi(u)$  we conclude that  $\psi(e) = 1$ . Further, it follows from the item (1) of Proposition 4 that  $\psi^2(e_k^p) = \psi(e_k^p)$ , i.e. for each value of the index  $k$  either  $\psi(e_k^p) = 0$ , or  $\psi(e_k^p) = 1$ . We find by calculating the value of  $\psi$  on each side of the equality (2) from Proposition 4 that  $\sum \psi(e_m^p) = 1$ . This is only possible if exactly one of the addends differs from zero. So, for the considered character  $\psi$  the set  $\{k : 0 \leq k < p\}$  contains a unique  $k$  such that the equality  $\psi(e_m^p) = \delta_{km}$  holds for any  $m$ . Let us compute the value of the functional  $\psi$  on an arbitrary function  $u \in \mathcal{A}_p$  using the equality (3) from Proposition 4:

$$\psi(u) = \sum u(m)\psi(e_m^p) = \sum u(m)\delta_{km} = u(k) = \delta_k(u).$$

Assume now that  $\psi = \delta_r$  where  $0 \leq r < p$ . Then  $u(k) = u(r)$  for every  $p$ -periodic function  $u$ . In particular, for  $u = \text{Res}_p$  we have

$$r = u(r) = u(k) = k. \quad \square$$

Let us underline an important property of the function  $\kappa$ .

**PROPOSITION 7.** *If the restrictions of a character  $\psi$  to the subalgebras  $\mathcal{A}_m$  and  $\mathcal{A}_n$  are non-trivial, then*

$$\kappa(m; \psi) \equiv \kappa(n; \psi) \pmod{(m \wedge n)}$$

where  $m \wedge n$  is the greatest common divisor of  $m$  and  $n$ .

**PROOF.** Set  $m \wedge n = d$ . As the function  $\text{Res}_d$  belongs to both  $\mathcal{A}_m$  and  $\mathcal{A}_n$ , we have

$$\begin{aligned} \text{Res}_d(\kappa(m; \psi)) &= \delta_{\kappa(m; \psi)}(\text{Res}_d) = \psi(\text{Res}_d) = \\ &= \delta_{\kappa(n; \psi)}(\text{Res}_d) = \text{Res}_d(\kappa(n; \psi)), \end{aligned}$$

as was to be proved.  $\square$

#### 4. The ring of characters

**4.1. Convolutions and reflection in  $\mathcal{G}$ .** Denote by  $\text{Sym}(\mathcal{A} \otimes \mathcal{A})$  the algebra consisting of all periodic (with respect to both arguments) functions  $w : \mathbb{Z}^2 \rightarrow \mathbb{C}$  such that  $w(x, y) = w(y, x)$ . If a function  $w$  is  $p$ -periodic with respect to  $x$ , it will be  $p$ -periodic with respect to  $y$  as well



(and conversely). Indeed,  $w(x, y+p) = w(y+p, x) = w(y, x) = w(x, y)$ . Let us agree to denote the value  $\phi(u)$  of a functional  $\phi \in \mathcal{A}^*$  by  $\phi_x u(x)$ . Thus, the variable  $x$  twice repeated in this expression is *umbral* i. e. the letter  $x$  can be replaced by any other one. Fixing the value of one of the arguments one can consider the function  $w$  as some element of  $\mathcal{A}$ .

DEFINITION 3. Define the *direct product*  $\phi \otimes \psi$  of two functionals  $\phi$  and  $\psi$  by the equality  $(\phi \otimes \psi)w = \phi_x \psi_y w(x, y)$ .

This is a linear functional on the algebra  $\text{Sym}(\mathcal{A} \otimes \mathcal{A})$ .

PROPOSITION 8. *The direct product is commutative.*

PROOF. Let  $w$  be an arbitrary  $p$ -periodic function taken from the algebra  $\text{Sym}(\mathcal{A} \otimes \mathcal{A})$ . Twice applying the equality (3) from Proposition 4, one can represent this function as

$$(4) \quad w(x, y) = \sum_{j=0}^{p-1} w(j, y) e_j^p(x) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(j, k) e_j^p(x) e_k^p(y).$$

Let us apply the linear functionals  $\phi \otimes \psi$  and  $\psi \otimes \phi$  to both sides of this equality:

$$\begin{aligned} (\phi \otimes \psi)w &= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(j, k) \phi(e_j^p) \psi(e_k^p). \\ (\psi \otimes \phi)w &= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(j, k) \psi(e_j^p) \phi(e_k^p). \end{aligned}$$

By changing the index names in the second equality and using the symmetry of the function  $w$ , we find that

$$\begin{aligned} (\psi \otimes \phi)w &= \sum_{k=0}^{p-1} \sum_{j=0}^{p-1} w(k, j) \psi(e_k^p) \phi(e_j^p) = \\ &= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(k, j) \phi(e_j^p) \psi(e_k^p) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(j, k) \phi(e_j^p) \psi(e_k^p). \end{aligned}$$

Hence,  $\psi \otimes \phi = \phi \otimes \psi$ .  $\square$

We shall define operators  $s$  and  $p$  from  $\mathcal{A}$  into  $\text{Sym}(\mathcal{A} \otimes \mathcal{A})$  and an operator  $n$  from  $\mathcal{A}$  into  $\mathcal{A}$ . For any  $u \in \mathcal{A}$  we set

$$(su)(x, y) = u(x + y); \quad (pu)(x, y) = u(xy); \quad (nu)(x) = u(-x).$$

It is clear that these operators are multiplicative:

$$s(uv) = (su)(sv); \quad p(uv) = (pu)(pv); \quad n(uv) = (nu)(nv).$$

DEFINITION 4. We call the functionals  $\phi \oplus \psi = (\phi \otimes \psi)_s$  and, respectively,  $\phi \odot \psi = (\phi \otimes \psi)_p$  a *plus-* and a *dot-convolutions* of  $\phi$  and  $\psi$ . The functional  $\ominus \phi = \phi_n$  is called a *reflection* of  $\phi$ .

#### 4.2. Ring properties of the operations in $\mathcal{G}$ .

PROPOSITION 9. *Both convolutions in the previous definition are commutative and associative.*

PROOF. The commutativity of operations is an obvious corollary from Proposition 8. The associativity of the dot-convolution follows from the equalities

$$\begin{aligned} (\phi \odot (\chi \odot \psi))(u) &= \phi_x(\chi \odot \psi)_s u(xs) = \phi_x \chi_y \psi_z u(x(yz)) \\ ((\phi \odot \chi) \odot \psi)(u) &= (\phi \odot \chi)_s \psi_z u(sz) = \phi_x \chi_y \psi_z u((xy)z) \end{aligned}$$

and from the associativity of the multiplication in  $\mathbb{Z}$ . The associativity of the plus-convolution can be established quite similarly.  $\square$

PROPOSITION 10. *The functionals  $\theta = \delta_0$  and  $\varepsilon = \delta_1$  are neutral elements of the operations  $\oplus$  and  $\odot$  respectively.*

PROOF.

$$\begin{aligned} (\phi \oplus \theta)(u) &= \phi_x \theta_y u(x+y) = \phi_x u(x+0) = \phi_x u(x) = \phi(u), \\ (\phi \odot \varepsilon)(u) &= \phi_x \varepsilon_y u(xy) = \phi_x u(x \cdot 1) = \phi_x u(x) = \phi(u), \end{aligned}$$

i. e.  $\theta$  and  $\varepsilon$  — are neutral elements of the corresponding convolutions.  $\square$

PROPOSITION 11. *The direct product of characters of the algebra  $\mathcal{A}$  is a character of the algebra  $\text{Sym}(\mathcal{A} \otimes \mathcal{A})$ .*

PROOF. For any two functions  $u$  and  $v$  from the algebra  $\mathcal{A}$ , in view of the multiplicativity of the functionals  $\phi$  and  $\psi$ , we have

$$\begin{aligned} (\phi \otimes \psi)(uv) &= \phi_x \psi_y u(x, y) v(x, y) = \phi_x((\psi_y u(x, y))(\psi_y v(x, y))) = \\ &= \phi_x \psi_y u(x, y) \phi_x \psi_y v(x, y) = (\phi \otimes \psi)(u)(\phi \otimes \psi)(v). \end{aligned}$$

The multiplicativity of the functional  $\phi \otimes \psi$  is thus proved.  $\square$

PROPOSITION 12. *The set  $\mathcal{G}$  of all characters of the algebra  $\mathcal{A}$  is closed under the binary operations  $\oplus$  and  $\odot$ , and under the unary operation  $\ominus$  of the reflection.*

PROOF. If  $\phi$  and  $\psi$  are elements of  $\mathcal{G}$  then, by the multiplicativity of the operator  $s$  and according to Proposition 11, we have

$$\begin{aligned} (\phi \oplus \psi)(uv) &= ((\phi \otimes \psi)_s)(uv) = (\phi \otimes \psi)(su sv) = \\ &= (\phi \otimes \psi)(su)(\phi \otimes \psi)(sv) = (\phi \oplus \psi)(u)(\phi \oplus \psi)(v). \end{aligned}$$

Analogously, by the multiplicativity of the operator  $p$ , we have

$$\begin{aligned} (\phi \odot \psi)(uv) &= ((\phi \otimes \psi)p)(uv) = (\phi \otimes \psi)(pu\,pv) = \\ &= (\phi \otimes \psi)(pu)(\phi \otimes \psi)(pv) = (\phi \odot \psi)(u)(\phi \odot \psi)(v). \end{aligned}$$

The multiplicativity of the functionals  $\phi \oplus \psi$  and  $\phi \odot \psi$ , and the closeness of  $\mathcal{G}$  under both convolutions is thus proved. Since the operator  $n$  is also multiplicative, for any character  $\phi$  we obtain

$$(\ominus \phi)(uv) = (\phi n)(uv) = \phi(nu\,nv) = \phi(nu)\phi(nv) = (\ominus \phi)(u)(\ominus \phi)(v).$$

Consequently, the functional  $\ominus \phi$  is a character.  $\square$

**PROPOSITION 13.** *The set  $\mathcal{G}$  of all characters of the algebra  $\mathcal{A}$  is a commutative ring with the operation of addition  $\oplus$ , multiplication  $\odot$ , zero  $\theta$  and the unit element  $\varepsilon$ .*

**PROOF.** Taking into account Propositions 9, 10, 12 it remains to prove the existence of an opposite for any character  $\chi$  and to verify the distributivity

$$(5) \quad (\phi \oplus \psi) \odot \chi = (\phi \odot \chi) \oplus (\psi \odot \chi).$$

Let us show that the additive convolution of a character  $\chi$  with another character  $\ominus \chi$  results in an additively neutral character  $\theta$ . Indeed, for an arbitrary periodic function  $u$  we have

$$(\chi \oplus (\ominus \chi))(u) = \chi_x(\ominus \chi)_y u(x+y) = \chi_x \chi_y u(x-y).$$

Let  $p$  be the period of the function  $u$ . As it follows from Proposition 6, there exists an integer  $k_p = \kappa(p; \chi)$  such that for the arbitrary element  $v$  from  $\mathcal{A}_p$  the value of the functional  $\chi$  is  $v(k_p)$ . Hence

$$\chi_x \chi_y u(x-y) = \chi_x u(x - k_p) = u(k_p - k_p) = u(0) = \theta(u).$$

Thus,  $\chi \oplus (\ominus \chi) = \theta$ . Let us verify the distributivity for an arbitrary function  $u \in \mathcal{A}_p$ . Applying first the left-hand side of the equality (5) to  $u$ :

$$\begin{aligned} ((\phi \oplus \psi) \odot \chi)(u) &= (\phi \oplus \psi)_s \chi_t u(st) = (\phi \oplus \psi)_s u(s\kappa(p; \chi)) = \\ &= \phi_x \psi_y u((x+y)\kappa(p; \chi)) = u((\kappa(p; \phi) + \kappa(p; \psi))\kappa(p; \chi)), \end{aligned}$$

and then its right-hand side:

$$\begin{aligned} (\phi \odot \chi) \oplus (\psi \odot \chi)(u) &= (\phi \odot \chi)_s (\psi \odot \chi)_t u(s+t) = \\ &= (\phi \odot \chi)_s \psi_y \chi_z u(s+yz) = (\phi \odot \chi)_s u(s + \kappa(p; \psi)\kappa(p; \chi)) = \\ &= \phi_x \chi_z u(xz + \kappa(p; \psi)\kappa(p; \chi)) = u(\kappa(p; \phi)\kappa(p; \chi) + \kappa(p; \psi)\kappa(p; \chi)), \end{aligned}$$

we see that the relation (5) holds.  $\square$

DEFINITION 5. In what follows, the binary operation  $\phi \oplus (\ominus \psi)$  will be written as  $\phi \ominus \psi$ .

## 5. Topological ring $(\mathcal{G}, \gamma)$ .

**5.1. Cluster topology in the ring  $\mathcal{G}$ .** Let  $p$  be an arbitrary positive integer.

DEFINITION 6. We shall say that elements  $\phi$  and  $\psi$  of the ring of characters are  $p$ -equivalent if the functionals  $\phi$  and  $\psi$  coincide on the subalgebra  $\mathcal{A}_p$ . Denote by  $V_p(\psi)$  the  $p$ -equivalence class containing the character  $\psi$ . These classes will be called *clusters*.

PROPOSITION 14. *The character  $\psi$  is the unique element which belongs to all clusters  $V_n(\psi)$ .*

PROOF. Let  $\phi \in V_n(\psi)$  for all positive integers  $n$ . Let us show that  $\phi = \psi$ . Since each function  $u$  from  $\mathcal{A}$  is periodic, it belongs to a subalgebra  $\mathcal{A}_n$ . But as  $\phi|_{\mathcal{A}_n} = \psi|_{\mathcal{A}_n}$ , we have  $\phi(u) = \psi(u)$ . As  $p$  is an arbitrary function, the functionals  $\phi$  and  $\psi$  coincide on the whole algebra  $\mathcal{A}$ .  $\square$

PROPOSITION 15. *For any character  $\psi$  and positive integers  $m$  and  $n$  we have  $V_{m \vee n}(\psi) \subseteq V_m(\psi) \cap V_n(\psi)$ . where  $m \vee n$  is the least common multiple of  $m$  and  $n$ .*

PROOF. It is obvious that every  $m$ -periodic function  $u$  is also  $(m \vee n)$ -periodic. Therefore the coincidence of the functionals  $\phi$  and  $\psi$  on the subalgebra  $\mathcal{A}_{m \vee n}$  implies their coincidence on the subalgebra  $\mathcal{A}_m \subseteq \mathcal{A}_{m \vee n}$ . In other words,  $V_{m \vee n}(\psi) \subseteq V_m(\psi)$ . Analogously,  $V_{m \vee n}(\psi) \subseteq V_n(\psi)$ .  $\square$

Propositions 14 and 15 mean that the system of clusters  $V_n(\psi)$  can be considered as a base of neighborhoods of the point  $\psi \in \mathcal{G}$  which defines a Hausdorff topology on  $\mathcal{G}$ .

DEFINITION 7. A topology defined by the base of neighborhoods  $V_n(\psi)$  in the ring  $\mathcal{G}$  will be called a *cluster topology*.

In the following we will denote the clusters  $V_n(\theta)$  by  $V_n$ .

PROPOSITION 16. *Any cluster  $V_n$  is a symmetric subset of  $\mathcal{G}$ .*

PROOF. We have to show that if  $\phi$  belongs to  $V_n$  then  $\ominus \phi$  belongs to  $V_n$  as well. The cluster  $V_n$  consists of the elements of the ring  $\mathcal{G}$  which act as the zero element on  $n$ -periodic functions:  $\phi(u) = \theta(u) = u(0)$ . Hence for these functions we have  $(\ominus \phi)(u) = \phi_x u(-x) = u(0)$  and  $\ominus \phi \in V_n$ .  $\square$

PROPOSITION 17. *Each cluster  $V_n$  is a subgroup in the additive group of the ring  $\mathcal{G}$ .*

PROOF. In view of Proposition 16, it is sufficient to show that the cluster  $V_n$  is closed under additive convolution. Let  $\phi$  and  $\psi$  be elements of  $V_n$ . For any  $n$ -periodic function  $u$  we have

$$(\phi \oplus \psi)(u) = \phi_x \psi_y u(x+y) = \phi_x u(x+0) = u(0),$$

so that  $\phi \oplus \psi \in V_n$ .  $\square$

PROPOSITION 18. *The cluster  $V_n$  is an ideal of the ring  $\mathcal{G}$ .*

PROOF. Let  $\phi$  be an arbitrary character, and let  $\psi$  be an element of the cluster  $V_n$ . For each  $n$ -periodic function  $u$  we have

$$(\phi \odot \psi)(u) = \phi_x \psi_y u(xy) = \phi_x u(0)e(x) = u(0)\phi(e) = u(0),$$

so  $\phi \odot \psi$  is an element of  $V_n$ .  $\square$

PROPOSITION 19.  *$V_n(\psi)$  is a  $\psi$ -translation of the cluster  $V_n$ .*

PROOF. Recall that the  $\psi$ -translation  $(V_n)_\psi$  of the subset  $V_n \subseteq \mathcal{G}$  is defined by the equality  $(V_n)_\psi = \{\psi \oplus \eta : \eta \in V_n\}$ . Hence  $\phi \in (V_n)_\psi$  if and only if  $(\phi \ominus \psi)|_{\mathcal{A}_n} = \theta|_{\mathcal{A}_n}$ . Since  $V_n$  is symmetric, without loss of generality we can consider  $0 \leq \kappa(n; \psi) \leq \kappa(n; \phi) \leq n-1$ . The fact that  $\phi$  belongs to the set  $(V_n)_\psi$  is equivalent to the fact that for any function  $u$  from  $\mathcal{A}_n$

$$u(\kappa(n; \phi) - \kappa(n; \psi)) = \phi_x \psi_y u(x-y) = (\phi \ominus \psi)(u) = \theta(u) = u(0).$$

Since the subalgebra  $\mathcal{A}_n$  separates the points  $0, \dots, n-1$ , the values  $\kappa(n; \phi)$  and  $\kappa(n; \psi)$  coincide, which is equivalent to the coincidence of the characters  $\phi$  and  $\psi$  on the subalgebra  $\mathcal{A}_n$ , i.e., to the condition  $\phi \in V_n(\psi)$ .  $\square$

PROPOSITION 20. *The cluster topology is compatible with the ring structure of  $\mathcal{G}$ .*

PROOF. As noted in Proposition 19, the base of neighborhoods of an arbitrary point  $\psi$  from the ring  $\mathcal{G}$  consists of the zero neighborhoods translations which, by Proposition 16, are symmetric subsets of the ring  $\mathcal{G}$ . It remains to prove that the clusters  $V_n$  satisfy the following conditions from Section 1.2:

- (1) The zero of the ring  $\mathcal{G}$  is the unique element which belongs to all  $V_n$ .
- (2) The intersection  $V_m \cap V_n$  includes some neighborhood of zero.
- (3) For each neighborhood  $V_n$  there exists a neighborhood  $V_m$  such that  $\{\phi \oplus \psi : \phi, \psi \in V_m\} \subseteq V_n$ .

- (4) For each neighborhood  $V_n$  and for any  $\psi \in \mathcal{G}$  one can find a neighborhood  $V_m$  such that  $\{\phi \odot \psi : \phi \in V_m\} \subseteq V_n$ .
- (5) For any neighborhood  $V_n$  there exists a neighborhood  $V_m$  such that  $\{\phi \odot \psi : \phi \odot \psi \in V_m\} \subseteq V_n$ .

It follows from Proposition 14 that the character  $\theta$  is the only element belonging to all clusters  $V_n$ , while Proposition 15 implies the condition (2):  $V_{m \vee n} \subseteq V_m \cap V_n$ . Next, in view of Proposition 17,  $V_n$  is a subgroup in the additive group of the ring  $\mathcal{G}$ , therefore the condition (3) is satisfied, for instance, if  $m = n$ . Finally, (4) and (5), for  $m = n$ , follow from the fact that, in view of Proposition 18,  $V_n$  is an ideal of the commutative ring  $\mathcal{G}$ .  $\square$

**5.2. Cluster topology as the Gelfand topology.** Let us correlate the cluster topology and the Gelfand topology  $\gamma$  which, according to (3), can be determined by the subbase of the neighborhoods of the points  $\psi \in \mathcal{G}$ :

$$U_R(u; \psi) = \{\phi \in \mathcal{G} : |\phi(u) - \psi(u)| < R\} \quad (u \in \mathcal{A}, R > 0).$$

PROPOSITION 21. *The cluster  $V_n(\psi)$  can be represented in the form*

$$(6) \quad V_n(\psi) = \bigcap_{j=0}^{n-1} U_R(e_j^n; \psi),$$

where  $R$  is an arbitrary number from the interval  $(0; 1)$ , and the functions  $e_j^n$  are indicators of the rasters  $R_n(j)$  considered in Proposition 4.

PROOF. First we show that  $V_n(\psi)$  belongs to every neighborhood  $U_R(e_j^n; \psi)$  for any positive  $R$ . Since  $e_j^n$  are elements of the subalgebra  $\mathcal{A}_n$ , where the functionals from  $V_n(\psi)$  coincide, then for any  $\phi \in V_n(\psi)$  we have

$$\phi \in \{\phi : |\phi(e_j^n) - \psi(e_j^n)| = 0\} \subseteq \{\phi : |\phi(e_j^n) - \psi(e_j^n)| < R\} = U_R(e_j^n; \psi).$$

Thus, the right-hand side of the equality (6) includes the left-hand one. Now we prove the converse inclusion. Suppose that character  $\phi$  belongs to all the neighborhoods  $U_R(e_j^n; \psi)$ . According to Proposition 6, its contraction to the subalgebra  $\mathcal{A}_n$  has the form  $\phi(u) = u(\kappa(n; \phi))$ , where  $0 \leq \kappa(n; \phi) \leq n - 1$ . Consequently,

$$\phi(e_j^n) = e_j^n(\kappa(n; \phi)) = \delta_{j\kappa(n; \phi)}.$$

Since  $\psi$  belongs to every neighborhood  $U_R(e_j^n; \psi)$ , then  $\psi(e_j^n) = \delta_{j\kappa(n; \psi)}$ . Thus, for all  $0 \leq j \leq n - 1$  the following inequalities hold:

$$|\delta_{j\kappa(n; \phi)} - \delta_{j\kappa(n; \psi)}| = |\phi(e_j^n) - \psi(e_j^n)| < R \leq 1,$$

which mean that for any  $j$  the integer  $\delta_j \kappa(n; \phi) - \delta_j \kappa(n; \psi)$  can only be zero. It follows that  $\kappa(n; \phi) = \kappa(n; \psi)$ , and therefore

$$\phi(u) = u(\kappa(n; \phi)) = u(\kappa(n; \psi)) = \psi(u)$$

on all  $u$  from  $\mathcal{A}_n$  and character  $\phi$  belongs to the cluster  $V_n(\psi)$ .  $\square$

**PROPOSITION 22.** *The Gelfand topology  $\gamma$  in the ring of characters  $\mathcal{G}$  coincides with the cluster topology and thus is consistent with the ring structure of  $\mathcal{G}$ .*

**PROOF.** In the Gelfand topology  $\gamma$ , finite intersections of the type  $U = \bigcap U_{R_j}(u_j; \psi)$  form a basis of neighborhoods of the point  $\psi$ , where  $u_j$  are arbitrary periodic functions on  $\mathbb{Z}$ , and  $R_j$  are arbitrary positive numbers. The equality (6) shows that each cluster  $V_n(\psi)$  belongs to the base of neighborhoods of the point  $\psi$ . Consequently, the cluster topology is weaker than the topology  $\gamma$ . Conversely, let  $p$  be the common period of the functions  $u_j$  which form the neighborhood  $U$  of the element  $\psi$ . For  $\phi$  from the cluster  $V_p$  we have

$$\phi \in \{\phi : |\phi(u_j) - \psi(u_j)| = 0\} \subseteq \{\phi : |\phi(u_j) - \psi(u_j)| < R_j\} = U_{R_j}(u_j; \psi)$$

for all values of the index  $j$ , so that  $V_p \subseteq U$ , and the topology  $\gamma$  is weaker than the cluster topology.  $\square$

## 6. Introduction into polyadic analysis

**6.1. Ring  $\mathcal{P}$  of polyadic numbers.** Let  $\mathcal{C}$  be the commutative ring of all sequences  $\alpha : \mathbb{N} \rightarrow \mathbb{Z}$  with pointwise operations of addition and multiplication induced from  $\mathbb{Z}$ .

A constant  $\alpha$  coincides with the result of its *shift*  $S$  defined by the equality  $(S\alpha)_n = \alpha_{n+1}$ . Associating to every integer  $m$  the constant of the same value, we can consider the ring  $\mathbb{Z}$  as a subring of all constants from the ring  $\mathcal{C}$ . Without risk of confusion, we shall use the same notation for constants and their values.

As usually, a statement is said to be true *for almost all* positive integers  $n$  if it is false only for a finite number of  $n \in \mathbb{N}$ .

Let us introduce a congruence relation in the ring of sequences  $\mathcal{C}$ .

**DEFINITION 8.** We shall write  $\alpha \equiv \beta$  for sequences  $\alpha$  and  $\beta$  if for each positive integer  $n$  the congruences  $\alpha_k \equiv \beta_k \pmod{n}$  hold for almost all  $k \in \mathbb{N}$ .

In particular, this definition implies that any two sequences with a finite number of different elements are congruent, and a sequence is congruent with the null constant if for each positive integer  $n$  almost all its elements are divisible by  $n$ . This is an obvious analogue of

the fact that zero is divisible by any positive integer. Thus a sequence congruent with the null constant will be called a *0-sequence*. It is easy to see that 0-sequences form an ideal  $\mathcal{C}_0$  of the ring  $\mathcal{C}$  stable under the shift  $S$  which makes it possible to apply the shift operation  $S$  to elements of the quotient-ring  $\mathcal{C}/\mathcal{C}_0$ . In addition, the elements  $\mathcal{C}_0$  of the form  $\alpha p$  ( $p \in \mathbb{N}$ ) can be divided by  $p$ : if almost all components of the sequence  $\alpha p$  are divisible by an arbitrary preassigned  $np$ , then almost all components of the sequence  $\alpha$  are divisible by an arbitrary preassigned  $n$ .

DEFINITION 9. By *polyadic number* we will call any constant from the commutative ring  $\mathcal{C}/\mathcal{C}_0$ , i.e. any class  $\alpha$  of congruent sequences  $\alpha$  such that  $\alpha \equiv S\alpha$ . We will say that the sequence  $\alpha \in \alpha$  *represents* the polyadic number  $\alpha$ .

Polyadic numbers form a ring denoted by  $\mathcal{P}$ . The ring  $\mathcal{P}$  can be considered as a module over the ring  $\mathbb{Z}$  if we define the product  $m\alpha$  of a positive integer  $m$  by an arbitrary  $\alpha$  from  $\mathcal{P}$  as a class of all sequences congruent to sequences of the form  $m\alpha$  ( $\alpha \in \alpha$ ).

DEFINITION 10. By associating each positive integer  $m$  with the element  $\mathcal{Z}(m)$  of the ring  $\mathcal{P}$  containing the constant sequence with the value  $m$ , we obtain the *canonical embedding*  $\mathcal{Z} : \mathbb{Z} \rightarrow \mathcal{P}$ .

This embedding whose image will be denoted by  $\mathcal{Z}$  is a strict homomorphism of the rings.

DEFINITION 11. We will say that an element  $\mathcal{Z}(m)$  of the ring  $\mathcal{Z}$  is an *integer polyadic number with the value  $m$* .

PROPOSITION 23. *If a sequence  $\beta = n\alpha$  ( $n \in \mathbb{N}$ ) represents a polyadic number  $\beta$ , then the sequence  $\alpha$  represents a polyadic number as well.*

PROOF. Since  $n(\alpha - S\alpha) = \beta - S\beta$  is the 0-sequence and since in  $\mathcal{C}_0$  the division by  $n \in \mathbb{N}$  is admissible, then  $\alpha - S\alpha$  is the 0-sequence.  $\square$

## 6.2. Topological ring $(\mathcal{P}, \sigma)$ .

Let us associate each positive integer  $n$  with the principal ideal  $\mathcal{G}^{(n)}$  of the ring  $\mathcal{P}$  defined by the following equality

$$\mathcal{G}^{(n)} = \{n\alpha : \alpha \in \mathcal{P}\}.$$

PROPOSITION 24. *The zero of the ring  $\mathcal{P}$  is the unique element belonging to all ideals  $\mathcal{G}^{(n)}$ .*



PROOF. Let the class  $\beta$  be common to all  $\mathcal{G}^{(n)}$ , and let  $p$  be an arbitrary positive integer. Then, in particular,  $\beta \in \mathcal{G}^{(p)}$ . All components of a sequence  $\beta = \alpha p$  from the class  $\beta$  are divisible by  $p$ . Hence almost all components of any sequence from the class  $\beta$  are divisible by any arbitrary positive integer  $p$ , and thus  $\beta$  is zero of the ring  $\mathcal{P}$ .  $\square$

PROPOSITION 25. *If a positive integer  $p$  divides a positive integer  $n$  then  $\mathcal{G}^{(n)} \subseteq \mathcal{G}^{(p)}$ .*

PROOF. Let  $\gamma \in \mathcal{G}^{(n)}$ , i. e.  $\gamma = n\beta$ , where  $\beta$  is an element of the ring  $\mathcal{P}$ , and let  $n = pd$ . All elements of the sequence  $\gamma = n\beta = p(d\beta)$  from the class  $\gamma$  are divisible by  $p$ . Therefore, almost all elements of any sequence from the class  $\gamma$  are divisible by  $p$ , and  $\gamma \in \mathcal{G}^{(p)}$ .  $\square$

PROPOSITION 26. *For arbitrary positive integers  $m$  and  $n$  the following equality  $\mathcal{G}^{(m)} \cap \mathcal{G}^{(n)} = \mathcal{G}^{(m \vee n)}$  holds, where  $m \vee n$  is the least common multiple of the numbers  $m$  and  $n$ .*

PROOF. As  $m$  and  $n$  divide  $m \vee n$ , the inclusion  $\mathcal{G}^{(m \vee n)} \subseteq \mathcal{G}^{(m)} \cap \mathcal{G}^{(n)}$  follows from Proposition 25.

Let us verify the converse inclusion. Let  $\gamma \in \mathcal{G}^{(m)} \cap \mathcal{G}^{(n)}$ , then  $\gamma = m\alpha$ , and  $\gamma = n\beta$ , where  $\alpha, \beta$  are elements of the ring  $\mathcal{P}$ . Denote by  $d = m \wedge n$  the greatest common divisor of the numbers  $m$  and  $n$ . The following equalities hold:

$$(7) \quad m = dp, \quad n = dq \quad (\text{where } p \wedge q = 1), \quad m \vee n = pdq = qm = pn.$$

If the elements  $\alpha$  and  $\beta$  are represented by sequences  $\alpha$  and  $\beta$  respectively, then  $\alpha dp \equiv \beta dq$ . In view of Proposition 23,  $\alpha p \equiv \beta q$ , so that the difference  $\alpha_k p - \beta_k q$  is divisible by  $pq$  for almost all  $k$ . As  $p \wedge q = 1$ , for almost all  $k$  there are numbers  $\alpha'_k$  and  $\beta'_k$  such that  $\alpha_k = \alpha'_k q$ , and  $\beta_k = \beta'_k p$ . Arbitrarily extending the functions  $k \mapsto \alpha'_k$  and  $k \mapsto \beta'_k$  to the sequences  $\alpha'$  and  $\beta'$ , we obtain (taking into account equalities (7)) the representations of the class  $\gamma$  by the sequences  $\alpha' qm = \alpha'(m \vee n)$  and  $\beta' pn = \beta'(m \vee n)$   $\square$

We shall introduce a Hausdorff topology in the ring of polyadic numbers making it a topological ring.

PROPOSITION 27. *The ideals  $\mathcal{G}^{(n)}$  form a (countable) base of zero neighborhoods, defining in the ring  $\mathcal{P}$  a Hausdorff topology  $\sigma$  compatible with its algebraic structure.*

PROOF. A base of zero neighborhoods is defined in Section 1.2. Condition (1) holds by virtue of the Proposition 24. Condition (2) is provided by Proposition 7. The remaining conditions hold because  $\mathcal{G}^{(n)}$  are ideals: here one and the same set  $\mathcal{G}^{(n)}$  can be used for both sets  $U$  and  $V$ .  $\square$

## 7. Compactness of the ring $(\mathcal{P}, \sigma)$

### 7.1. Grids. Finite partitions of the space $(\mathcal{P}, \sigma)$ .

DEFINITION 12. We call by a *grid*  $\mathcal{G}(n; \alpha)$  with a *width*  $n \in \mathbb{N}$  and a *center*  $\alpha \in \mathcal{P}$  any  $\alpha$ -translation of the ideal  $\mathcal{G}^{(n)}$ , i.e. each subset of the ring  $\mathcal{P}$  having the form  $\mathcal{G}(n; \alpha) = \{\alpha + \gamma : \gamma \in \mathcal{G}^{(n)}\}$ .

For each point  $\alpha \in \mathcal{P}$  the grids  $\mathcal{G}(n; \alpha)$  form a base of its neighborhoods. Let us note some elementary properties of grids.

PROPOSITION 28. *If  $\beta \in \mathcal{G}(n; \alpha)$ , then  $\mathcal{G}(n; \alpha) = \mathcal{G}(n; \beta)$ .*

PROOF. Let  $\beta = \alpha + \gamma$ , where  $\gamma \in \mathcal{G}^{(n)}$ . Let us prove the inclusion  $\mathcal{G}(n; \alpha) \subseteq \mathcal{G}(n; \beta)$ . If  $\xi = \alpha + \eta$ , where  $\eta$  is an element of the ideal  $\mathcal{G}^{(n)}$  then  $\xi = \beta + (\eta - \gamma)$ , where the element  $\eta - \gamma$  also belongs to  $\mathcal{G}^{(n)}$ . Therefore,  $\xi \in \mathcal{G}(n; \beta)$ . The inverse inclusion can be proved analogously.  $\square$

PROPOSITION 29. *Each grid  $\mathcal{G}(n; \alpha)$  is open. In particular, the ideals  $\mathcal{G}^{(n)} = \mathcal{G}(n; \theta)$  are open in the topology  $\sigma$ .*

PROOF. By Proposition 28, the grid  $\mathcal{G}(n; \alpha)$  is a neighborhood  $\mathcal{G}(n; \beta)$  for any point  $\beta$  of this grid.  $\square$

PROPOSITION 30. *If  $\delta$  belongs to both grids  $\mathcal{G}(m; \alpha)$  and  $\mathcal{G}(n; \beta)$ , then  $\mathcal{G}(m; \alpha) \cap \mathcal{G}(n; \beta) = \mathcal{G}(m \vee n; \delta)$ .*

PROOF. In view of Proposition 28, we should prove that

$$\mathcal{G}(m; \delta) \cap \mathcal{G}(n; \delta) = \mathcal{G}(m \vee n; \delta),$$

i.e. that  $\delta$  are translations of the ideal  $\mathcal{G}^{(m \vee n)}$ , and the intersections  $\mathcal{G}^{(m)} \cap \mathcal{G}^{(n)}$  coincide. The latter statement immediately follows from Proposition 26.  $\square$

Thus, two arbitrary grids either do not intersect or their intersection is a grid as well.

PROPOSITION 31. *If  $\beta \in \mathcal{G}(p; \alpha)$  then the grid  $\mathcal{G}(pq; \beta)$  for any positive integer  $q$  is absorbed by the grid  $\mathcal{G}(p; \alpha)$ . Otherwise, the intersection of these grids is empty.*

PROOF. Let  $\beta \in \mathcal{G}(p; \alpha)$ . Since  $\beta \in \mathcal{G}(pq; \beta)$ , then by Proposition 30,  $\mathcal{G}(p; \alpha) \cap \mathcal{G}(pq; \beta) = \mathcal{G}(p \vee pq; \beta) = \mathcal{G}(pq; \beta)$ , therefore  $\mathcal{G}(pq; \beta) \subseteq \mathcal{G}(p; \alpha)$ .

Now let  $\beta \notin \mathcal{G}(p; \alpha)$  and let  $\delta$  be an element belonging to both grids. Then we have  $(\delta - \alpha) \in \mathcal{G}^{(p)}$  and  $(\delta - \beta) \in \mathcal{G}^{(pq)} \subseteq \mathcal{G}^{(p)}$ . Consequently,  $\beta - \alpha = (\delta - \alpha) - (\delta - \beta) \in \mathcal{G}^{(p)}$ , i.e.  $\beta \in \mathcal{G}(p; \alpha)$ , which contradicts our supposition of the existence of the element  $\delta$ .  $\square$

PROPOSITION 32. *The subring of integer polyadic numbers  $\mathcal{Z}$  is dense in the topological ring  $(\mathcal{P}, \sigma)$ .*

PROOF. Let us verify that each grid  $\mathcal{G}(n; \alpha)$  contains some element from the ring  $\mathcal{Z}$ . Let the polyadic number  $\alpha$  be represented by a sequence of integers  $\alpha$ . Then there exists  $m$  such that  $\alpha_{k+1} \equiv \alpha_k \pmod n$  for all  $k \geq m$ . By the transitivity of the congruence,  $\alpha_k \equiv \alpha_m \pmod n$  for all  $k \geq m$ . Let  $\zeta$  be an integer polyadic number represented by a constant sequence  $\zeta$  with the value  $\zeta_k = \alpha_m$ . For  $k \geq m$ ,

$$(\zeta - \alpha)_k = \zeta_k - \alpha_k = \alpha_m - \alpha_k \equiv 0 \pmod n,$$

hence  $\zeta - \alpha$  belongs to the grid  $\mathcal{G}^{(n)}$ , which means that the integer polyadic number  $\zeta$  belongs to the grid  $\mathcal{G}(n; \alpha)$ .  $\square$

In the ring  $\mathcal{P}$  the theorem of *the division with a remainder* holds.

PROPOSITION 33. *For a given  $n \in \mathbb{N}$  each polyadic number  $\alpha$  can be uniquely represented in the form  $\alpha = n\gamma(\alpha) + \varrho(\alpha)$ , where  $\gamma(\alpha) \in \mathcal{P}$ , and  $\varrho(\alpha)$  is an integer polyadic number with the value  $0 \leq r(\alpha) \leq n-1$ .*

PROOF. Let us prove the existence of this representation. In view of Proposition 32, the grid  $\mathcal{G}(n; \alpha)$  contains an element  $\zeta = \mathcal{Z}(m)$ . Let  $m = kn + r$ , where  $0 \leq r \leq n-1$ . Then we have

$$\zeta = \mathcal{Z}(nk + r) = n\mathcal{Z}(k) + \mathcal{Z}(r).$$

On the other hand, the fact that the element  $\zeta$  belongs to the grid  $\mathcal{G}(n; \alpha)$  implies  $\zeta = \alpha + n\beta$ , where  $\beta \in \mathcal{P}$ . Comparing these two representations of the element  $\zeta$ , we conclude that  $\alpha = n\gamma + \varrho$ , where  $\gamma = \mathcal{Z}(k) - \beta$ , and  $\varrho = \mathcal{Z}(r)$ . Now let us prove the uniqueness. Suppose that  $\alpha = n\gamma + \mathcal{Z}(r)$ , and  $\alpha = n\gamma' + \mathcal{Z}(r')$ , where  $0 \leq r \leq r' \leq n-1$ . Subtracting term by term one representation of the polyadic number  $\alpha$  from the other, we see that  $\mathcal{Z}(r' - r) = n(\gamma - \gamma')$ , i. e. non-negative constant sequence with the value  $r' - r < n$  and some integer sequence  $n(\gamma_k - \gamma'_k)$  represent the same polyadic number. This is only possible when both sequences are 0-sequences.  $\square$

DEFINITION 13. The integer non-negative number  $r(\alpha)$  defined in Proposition 33 will be called a *remainder* of the division of a polyadic number  $\alpha$  by a positive integer  $n$  and denoted by

$$r(\alpha) = \text{Res}_n(\alpha).$$

PROPOSITION 34. *For each positive integer  $N$  one can split the ring  $\mathcal{P}$  into  $N$  disjoint grids with width  $N$ :*

$$(8) \quad \mathcal{P} = \bigcup_{r=0}^{N-1} \mathcal{G}(N; \mathcal{Z}(r)).$$

PROOF. According to Proposition 33, an arbitrary polyadic number  $\alpha$  can be represented in the form  $\alpha = \beta + \mathcal{Z}(r)$ , where the number  $\beta$  belongs to the ideal  $\mathcal{G}^{(N)}$  and  $0 \leq r < N$ . In other words,  $\alpha$  belongs to  $\mathcal{G}(N; \mathcal{Z}(r))$ . Since this representation is unique, the grids corresponding to different values of  $r$  do not intersect.  $\square$

COROLLARY. *For any positive integer  $n$  and  $0 \leq k < n!$ , the grid  $\mathcal{G}(n!; \mathcal{Z}(k))$  split into disjoint grids with the width  $(n+1)!$ :*

$$(9) \quad \mathcal{G}(n!; \mathcal{Z}(k)) = \bigcup_{m=0}^n \mathcal{G}((n+1)!; \mathcal{Z}(k + n! \cdot m)).$$

PROOF. Let us set  $N = (n+1)!$  in the equality (8):

$$\mathcal{P} = \bigcup_{0 \leq r < (n+1)!} \mathcal{G}((n+1)!; \mathcal{Z}(r)).$$

Compare the intersections of the grid  $\mathcal{G}(n!; \mathcal{Z}(k))$  with the left- and right-hand sides of this new equality:

$$\mathcal{G}(n!; \mathcal{Z}(k)) = \bigcup_{0 \leq r < (n+1)!} \mathcal{G}(n!; \mathcal{Z}(k)) \cap \mathcal{G}((n+1)!; \mathcal{Z}(r)).$$

Here again the right-hand side is a sum of disjoint components. Setting  $p = n!$ ,  $q = n+1$ ,  $\alpha = \mathcal{Z}(k)$ ,  $\beta = \mathcal{Z}(r)$  in Proposition 31, we conclude that these components are nonempty only when  $\mathcal{Z}(r - k) \in \mathcal{G}^{(n!)}$ , and in this case they coincide with the grids  $\mathcal{G}((n+1)!; \mathcal{Z}(r))$ . It is easy to see that all numbers  $r$  from the interval  $0 \leq r < (n+1)!$  form a progression  $r = k + n! \cdot m$ , where  $0 \leq m \leq n$ .  $\square$

PROPOSITION 35. *Each grid  $\mathcal{G}(n; \alpha)$  is closed. In particular, the ideals  $\mathcal{G}^{(n)} = \mathcal{G}(n; \vartheta)$  are closed in the topology  $\sigma$ .*

PROOF. According to the equality (8), the number  $\alpha$  has to belong to some grid  $\mathcal{G}(n; \mathcal{Z}(k))$ . Hence, in view of Proposition 28,  $\mathcal{G}(n; \alpha) = \mathcal{G}(n; \mathcal{Z}(k))$ . Let us show that the grid  $\mathcal{G}(n; \mathcal{Z}(k))$  is closed. By Proposition 29, the grids  $\mathcal{G}(n; \mathcal{Z}(r))$  are open, and in view of the equality (8), the grid  $\mathcal{G}(n; \mathcal{Z}(k))$  is the complement of the (open) sum of the grids  $\mathcal{G}(n; \mathcal{Z}(r))$  where  $r \neq k$ . Therefore, the (coinciding) grids  $\mathcal{G}(n; \mathcal{Z}(k))$  and  $\mathcal{G}(n; \alpha)$  are closed.  $\square$

**7.2. Convergence. Hausdorff compact space  $(\mathcal{P}, \sigma)$ .** Convergence in the topological ring  $(\mathcal{P}, \sigma)$  is defined in the standard way.

DEFINITION 14. A sequence of polyadic numbers  $\alpha^{(n)}$  converges to the limit  $\alpha$ , if each neighborhood of zero  $\mathcal{G}^{(p)}$  contains almost all differences  $\alpha^{(n)} - \alpha$ .

It is clear, that the ideal  $\mathcal{G}^{(p)}$  has to contain almost all differences  $\alpha^{(n)} - \alpha^{(n+1)} = (\alpha^{(n)} - \alpha) - (\alpha^{(n+1)} - \alpha) \in \mathcal{G}^{(p)}$ . The converse is also true which follows from the proposition stated below.

**PROPOSITION 36.** *A sequence of polyadic numbers  $\alpha^{(n)}$  converges in  $(\mathcal{P}, \sigma)$  if and only if  $\alpha^{(n)} - \alpha^{(n+1)} \rightarrow \vartheta$ .*

**PROOF.** Let  $\alpha^{(n)} - \alpha^{(n+1)} \rightarrow \vartheta$ . Let us represent each polyadic number  $\alpha^{(n)}$  by an arbitrary integer sequence  $\alpha^{(n)}$ . By the definition of polyadic numbers, for any  $n$  one can find  $i(n)$  such that

$$(10) \quad k \geq i(n) \Rightarrow \alpha_k^{(n)} \equiv \alpha_{i(n)}^{(n)} \pmod{n!}.$$

Without loss of generality we can consider the sequence  $i(n)$  as *strictly increasing* so that  $i(n) \geq n$ . Let us show that the sequence  $\alpha_n = \alpha_{i(n)}^{(n)}$  represents some polyadic number  $\alpha$ . For an arbitrary positive integer  $p$  we verify that the difference  $\alpha_{i(n)}^{(n)} - \alpha_{i(n+1)}^{(n+1)}$  is divisible by  $p$  for almost all  $n$ . The condition  $\alpha^{(n)} - \alpha^{(n+1)} \rightarrow \vartheta$  implies the existence of  $n(p) \geq p$  such that for  $n \geq n(p)$  the differences  $\alpha^{(n)} - \alpha^{(n+1)}$  belong to the neighborhood of zero  $\mathcal{G}^{(p)}$ , i. e. for  $k \geq j(n)$  all remainders  $\alpha_k^{(n)} - \alpha_k^{(n+1)}$  are divisible by  $p$ . Now let  $k(n) = \sup\{i(n), i(n+1), j(n)\}$  and represent the difference  $\alpha_n - \alpha_{n+1} = \alpha_{i(n)}^{(n)} - \alpha_{i(n+1)}^{(n+1)}$  in the form

$$\left(\alpha_{i(n)}^{(n)} - \alpha_{k(n)}^{(n)}\right) + \left(\alpha_{k(n)}^{(n)} - \alpha_{k(n)}^{(n+1)}\right) + \left(\alpha_{k(n)}^{(n+1)} - \alpha_{i(n+1)}^{(n+1)}\right).$$

For  $n \geq n(p) \geq p$  the second summand is divisible by  $p$ , while the first and the last ones are divisible by  $n!$  and  $(n+1)!$ , respectively. Thus they both are divisible by  $p$ . Therefore, the above sequence  $\alpha_n$  indeed represents some polyadic number  $\alpha$ , and for  $k \geq n \geq n(p)$  all differences  $\alpha_n - \alpha_k$  are divisible by  $p$ . Let us now verify that  $\alpha^{(n)} \rightarrow \alpha$ , i. e. for each  $n \geq n(p)$  and  $k \geq i(n)$ , where the sequence  $i(n) \geq n$  is defined by the condition (10), the difference  $\alpha_k^{(n)} - \alpha_k$  is divisible by  $p$ . To this end, we write it in the form

$$(\alpha_n - \alpha_k) + \left(\alpha_k^{(n)} - \alpha_{i(n)}^{(n)}\right).$$

As has just been noted, the first summand is divisible by  $p$  and in view of (10), the second one is divisible by  $n!$ . By virtue of the inequality  $n \geq n(p) \geq p$ , the second summand is also divisible by  $p$ .  $\square$

**PROPOSITION 37.** *The ring  $(\mathcal{P}, \sigma)$  is a Hausdorff compact space.*

**PROOF.** In Proposition 27 the separability of the topology  $\sigma$  was proved. Suppose that  $(\mathcal{P}, \sigma)$  is not a compact. Suppose that the open

sets  $\mathcal{O}_\lambda$  form a covering  $\mathcal{P}$  where a finite subcovering cannot be found. Then among the two grids of the partition

$$\mathcal{P} = \bigcup_{r=0}^1 \mathcal{G}(2!; \mathcal{Z}(r)).$$

one can find a grid  $\mathcal{G}(2!; \mathcal{Z}(r_2))$ , which either cannot be covered with any finite sum of sets  $\mathcal{O}_\lambda$ , – otherwise, the whole space  $\mathcal{P}$  would have a finite covering of elements of the set  $\mathcal{O}_\lambda$ . By Proposition 31, the grid  $\mathcal{G}(2!; \mathcal{Z}(r_2))$  can be split into 3 disjoint grids

$$\mathcal{G}(2!; \mathcal{Z}(r_2)) = \bigcup_{r=0}^2 \mathcal{G}(3!; \mathcal{Z}(r_2 + 2! \cdot r)),$$

where at least one grid  $\mathcal{G}(3!; \mathcal{Z}(r_3))$  cannot be covered with a finite sum of sets  $\mathcal{O}_\lambda$  and so on. In this way we obtain a sequence of grids

$$\mathcal{G}(2!; \mathcal{Z}(r_2)) \supseteq \mathcal{G}(3!; \mathcal{Z}(r_3)) \supseteq \cdots \supseteq \mathcal{G}(k!; \mathcal{Z}(r_k)) \supseteq \cdots,$$

which cannot be covered with a finite number of elements from the set  $\mathcal{O}_\lambda$ . Since  $\mathcal{Z}(r_{k+1}) \in \mathcal{G}((k+1)!; \mathcal{Z}(r_{k+1})) \subseteq \mathcal{G}(k!; \mathcal{Z}(r_k))$ , the difference  $\mathcal{Z}(r_{k+1}) - \mathcal{Z}(r_k)$  belongs to the neighborhood of zero  $\mathcal{G}^{(k!)}$ . By Proposition 36, the sequence of integer polyadic numbers  $\mathcal{Z}(r_k)$  converges in  $(\mathcal{P}, \sigma)$  to some  $\alpha \in \mathcal{P}$ . Let  $\mathcal{O}$  is an element of the covering  $\mathcal{O}_\lambda$ , containing the point  $\alpha$ . Then  $\alpha$  belongs to the open set  $\mathcal{O}$  together with its neighborhood  $\mathcal{G}(n, \alpha)$ . Let  $m$  be such that  $\mathcal{Z}(r_k) \in \mathcal{G}(n, \alpha)$  for  $k \geq m$ . For  $k = \max\{m, n\}$  the number  $k!$  is divisible by  $n$ , so  $\mathcal{G}^{(k!)} \subseteq \mathcal{G}^{(n)}$  and  $\mathcal{G}(k!; \mathcal{Z}(r_k)) \subseteq \mathcal{G}(n; \mathcal{Z}(r_k))$ . As  $\mathcal{Z}(r_k) \in \mathcal{G}(n, \alpha)$ , according to Proposition 28,  $\mathcal{G}(n; \mathcal{Z}(r_k)) = \mathcal{G}(n, \alpha)$ , and the following inclusion holds:  $\mathcal{G}(k!; \mathcal{Z}(r_k)) \subseteq \mathcal{G}(n; \mathcal{Z}(r_k)) = \mathcal{G}(n, \alpha) \subseteq \mathcal{O}$ , which contradicts our assumption.  $\square$

## 8. Stabilizers

The notion of stabilizer is an interpretation of the polyadic numbers which allows one to relate the polyadic topological ring to the topological ring of characters.

### 8.1. Basic definitions.

**DEFINITION 15.** We shall say that the sequence  $\alpha : \mathbb{N} \rightarrow \mathbb{Z}$  *stabilizes* a function  $u : \mathbb{Z} \rightarrow \mathbb{C}$  at the *final value*  $v = (u \circ \alpha)_\infty$  if for almost all  $n$  the equality  $u(\alpha_n) = v$  holds.

**DEFINITION 16.** A sequence  $\alpha : \mathbb{N} \rightarrow \mathbb{Z}$  is said to be a *stabilizer* if it stabilizes each function  $u \in \mathcal{A}$ .

It follows from these definitions that for each periodic function  $u$  one can choose an integer  $n$  such that the final value  $(u \circ \alpha)_\infty$  will be  $u(n)$ . A “uniformity” condition restricts the class of stabilizers.

DEFINITION 17. A stabilizer  $\alpha$  is *absolute* if there exists an integer  $n$  such that for all  $u$  belonging to  $\mathcal{A}$  the following condition holds:

$$(u \circ \alpha)_\infty = u(n).$$

PROPOSITION 38. *For any absolute stabilizer  $\alpha$  there is a unique integer  $n$  which satisfies the condition  $(u \circ \alpha)_\infty = u(n)$  (uniformly with respect to  $u$  from  $\mathcal{A}$ ).*

PROOF. Let  $\alpha$  be an absolute stabilizer, and let  $p$  be an arbitrary positive integer. Let  $u = \text{Res}_p$ . By definition, there exists an integer  $n$  such that the final value  $(u \circ \alpha)_\infty$  is equal to  $u(n)$ . If there is another number  $n'$  which has the same property, then  $\text{Res}_p(n') = \text{Res}_p(n)$  and the remainder  $n' - n$  is divisible by any positive integer  $p$ .  $\square$

The above proposition allows us to introduce a function  $\nu$  which is well-defined on the set of all absolute stabilizers  $\alpha$  by the equality  $(u \circ \alpha)_\infty = u(\nu(\alpha))$ .

DEFINITION 18. The integer  $\nu(\alpha)$  will be called the *value* of an absolute stabilizer  $\alpha$ .

To prove that the mapping  $\nu$  is surjective it is sufficient to consider the constant sequence with the appropriate value.

DEFINITION 19. Absolute stabilizers with zero value will be called *zero stabilizers*.

Thus, a zero stabilizer  $\alpha$  can be described as a sequence of integers which stabilizes each function  $u$  from  $\mathcal{A}$  at the value  $u(0)$ . Weakening this condition by exchanging the algebra  $\mathcal{A}$  for its finite-dimensional subalgebra  $\mathcal{A}_p$ , we come to the concept of a *prezero* stabilizer.

DEFINITION 20. A stabilizer  $\alpha$  is said to be a *p-prezero* stabilizer if  $(u \circ \alpha)_\infty = u(0)$  for any  $p$ -periodic function  $u$ .

**8.2. Stabilizers and polyadic numbers.** Let us describe the concepts given in the previous section in terms of polyadic numbers.

PROPOSITION 39. *A sequence  $\alpha$  is a stabilizer if and only if it represents a polyadic number.*

PROOF. Let  $\alpha$  be a stabilizer. Then for any periodic function  $u$  we have  $u(\alpha_{k+1}) = u(\alpha_k)$  for almost all  $k$ . In particular, for an arbitrary positive integer  $p$  we have

$$\operatorname{Res}_p(\alpha_{k+1}) = \operatorname{Res}_p(\alpha_k),$$

for almost all  $k$ , i.e.  $\alpha_{k+1} \equiv \alpha_k \pmod p$  for almost all  $k$ , and the sequence  $\alpha$  represents a polyadic number.

Conversely, let  $\alpha \in \mathcal{P}$ . Choosing an arbitrary function  $u \in \mathcal{A}_p$ , let us verify that the sequence  $\alpha$  stabilizes  $u$ . Denote  $\alpha'_k = \alpha_{k+1} - \alpha_k$ . As the sequence  $\alpha$  represents a polyadic number,  $\alpha'$  is a 0-sequence, and beginning from  $k = n$  the equalities  $\alpha'_k = pq_k$  hold. Since  $u$  is a  $p$ -periodic function, we have for  $N \geq n$

$$u(\alpha_N) = u\left(\alpha_n + \sum_{k=n}^{N-1} \alpha'_k\right) = u\left(\alpha_n + p \sum_{k=n}^{N-1} q_k\right) = u(\alpha_n).$$

Consequently,  $\alpha$  is a stabilizer.  $\square$

PROPOSITION 40. *The class of zero stabilizers coincides with the class of 0-sequences representing the zero element  $\mathfrak{0}$  of the commutative ring  $\mathcal{P}$ .*

PROOF. Let  $\alpha$  be a zero stabilizer. Then for any periodic function  $u$  almost all numbers  $u(\alpha_k)$  are equal to  $u(0)$ . In particular, for any  $p$  the relation

$$\operatorname{Res}_p(\alpha_n) = \operatorname{Res}_p(0) = 0.$$

holds for almost all positive integers  $k$ . Therefore,  $\alpha \in \mathcal{C}_0$ , since  $\alpha_k \equiv 0$  for almost all  $k$ .

Conversely, let  $\alpha \in \mathcal{C}_0$ , i.e. almost all elements of the sequence  $\alpha$  are divisible by any preassigned positive integer. For an arbitrary  $p$ -periodic function  $u$ , let us verify that the final value of  $u$  exists and is equal to  $u(0)$ . Indeed, for  $k \geq n(p)$  the equalities  $\alpha_k = pq_k$  hold and in view of  $p$ -periodicity,  $u(\alpha_k) = u(pq_k) = u(0)$ .  $\square$

Using the notion of a zero stabilizer, it is easy to describe the class of absolute stabilizers.

PROPOSITION 41. *A sequence is an absolute stabilizer if and only if it represents an integer polyadic number.*

PROOF. Let  $\alpha$  be an absolute stabilizer. Let us introduce a constant sequence  $\alpha^c$  with the value  $\nu(\alpha)$ , and define the sequence  $\alpha^o$  as the remainder  $\alpha - \alpha^c$ . Let us verify that  $\alpha^o$  is a zero stabilizer. For an arbitrary function  $u \in \mathcal{A}$ , we define the periodic function  $v$  by the



equality  $v(\cdot) = u(\cdot - \nu(\alpha))$ . Since  $\alpha$  is an absolute stabilizer with the value  $\nu(\alpha)$ , the sequence  $\alpha^o$  stabilizes  $u$  at the final value  $u(0)$ :

$$(u \circ \alpha^o)_\infty = (v \circ \alpha)_\infty = v(\nu(\alpha)) = u(\nu(\alpha) - \nu(\alpha)) = u(0).$$

Thus  $\alpha^o$  is a zero stabilizer, and by Proposition 40, it is a 0-sequence, which implies that  $\alpha$  is an integer polyadic number with the value  $\nu(\alpha)$ .

Conversely, suppose that  $\alpha$  represents an integer polyadic number with the value  $m$ , i.e.  $\alpha_k = m + \alpha_k^o$ , where  $\alpha^o$  is a 0-sequence and (by Proposition 40) a zero stabilizer as well. Let us verify that  $\alpha$  is an absolute stabilizer with the value  $m$ . For an arbitrary function  $u$  from  $\mathcal{A}$ , we define the function  $v$  by the equality  $v(\cdot) = u(\cdot + m)$ . As  $\alpha^o$  is a zero stabilizer, the sequence  $\alpha$  stabilizes the function  $u$  at the final value  $(u \circ \alpha)_\infty = (v \circ \alpha^o)_\infty = v(0) = u(m)$ .  $\square$

**PROPOSITION 42.** *A sequence is a  $p$ -prezero stabilizer if and only if it represents an element of the ideal  $\mathcal{G}^{(p)}$ .*

**PROOF.** Let  $\alpha$  be a  $p$ -prezero stabilizer. By Proposition 39, it represents a polyadic number  $\alpha$ . Let us show that  $\alpha$  belongs to the ideal  $\mathcal{G}^{(p)}$ . Indeed, by the definition of a  $p$ -prezero stabilizer, for almost all  $k$  we have

$$\text{Res}_p(\alpha_k) = \text{Res}_p(0) = 0,$$

hence almost all elements of the sequence have the form  $\alpha_k = p\beta_k$ , where  $\beta$  is some sequence of integers. According to Proposition 23, the sequence  $\beta$  represents a polyadic number  $\beta$ . Therefore,  $\alpha$  represents the polyadic number  $\alpha = p\beta$ , belonging to the ideal  $\mathcal{G}^{(p)}$ .

Conversely, suppose that a sequence  $\alpha$  represents a polyadic number  $\alpha$  which belongs to the ideal  $\mathcal{G}^{(p)}$ , i.e., for almost all  $k$  elements of the sequence have the form  $\alpha_k = p\beta_k$ , where  $\beta$  is some sequence of integers. By Proposition 39, the sequence  $\alpha$  is a stabilizer. Let us verify that this stabilizer is  $p$ -prezero. In view of  $p$ -periodicity, for  $u \in \mathcal{A}_p$  the equalities  $u(\alpha_k) = u(p\beta_k) = u(0)$  hold for almost all  $k$ .  $\square$

## 9. Isomorphism of the rings $(\mathcal{P}, \sigma)$ and $(\mathcal{G}, \gamma)$

**9.1. Algebraic isomorphism of the rings  $\mathcal{P}$  and  $\mathcal{G}$ .** Let  $\mathcal{S}$  be a functional ring of all stabilizers with pointwise operations induced from  $\mathbb{C}$ . Let us associate each stabilizer  $\alpha$  from  $\mathcal{S}$  with the functional  $\pi\alpha : \mathcal{A} \rightarrow \mathbb{C}$  defined by the equality

$$(11) \quad (\pi\alpha)(u) = (u \circ \alpha)_\infty.$$

PROPOSITION 43. *For any stabilizer  $\alpha$  the functional  $\pi\alpha$  is a character.*

PROOF. Let  $e$  be the unit of the algebra  $\mathcal{A}$ , i. e.  $e$  is a constant function on  $\mathbb{Z}$  with the value 1. Since for any stabilizer  $\alpha$  the composition  $e \circ \alpha$  is a constant sequence with the value 1, then  $(\pi\alpha)(e) = (e \circ \alpha)_\infty = 1$  and the functional  $\pi\alpha$  is nonzero. For  $z \in \mathbb{C}$  and any function  $u \in \mathcal{A}$  we have

$$(\pi\alpha)(zu) = ((zu) \circ \alpha)_\infty = (z(u \circ \alpha))_\infty = z(u \circ \alpha)_\infty = z(\pi\alpha)(u),$$

which proves the homogeneity of the functional  $\pi\alpha$ . Using the distributive properties of the composition  $(u + v) \circ \alpha = u \circ \alpha + v \circ \alpha$  and  $(uv) \circ \alpha = (u \circ \alpha)(v \circ \alpha)$ , it is easy to verify its additivity

$$\begin{aligned} (\pi\alpha)(u + v) &= ((u + v) \circ \alpha)_\infty = (u \circ \alpha + v \circ \alpha)_\infty = \\ &= (u \circ \alpha)_\infty + (v \circ \alpha)_\infty = (\pi\alpha)(u) + (\pi\alpha)(v) \end{aligned}$$

and multiplicativity

$$\begin{aligned} (\pi\alpha)(uv) &= ((uv) \circ \alpha)_\infty = ((u \circ \alpha)(v \circ \alpha))_\infty = \\ &= (u \circ \alpha)_\infty (v \circ \alpha)_\infty = (\pi\alpha)(u)(\pi\alpha)(v). \end{aligned}$$

Thus, we have all the properties of characters.  $\square$

The above proposition allows us to introduce a mapping  $\pi$  acting from the ring  $\mathcal{S}$  into the ring  $\mathcal{G}$  by the rule  $\pi : \alpha \mapsto \pi\alpha$ .

PROPOSITION 44. *The mapping  $\pi$  is a homomorphism of the ring  $\mathcal{S}$  into the ring  $\mathcal{G}$ , whose kernel  $\ker \pi$  is the ideal  $\mathcal{C}_0$  of zero stabilizers.*

PROOF. First we prove that the mapping  $\pi$  is additive. It is sufficient to verify that the values of the functionals  $\pi(\alpha + \beta)$  and  $\pi\alpha \oplus \pi\beta$  are equal for any periodic function  $u$ . Let  $p$  be the period of the function  $u$ . Let us quote the equality (4) which holds for any function  $w$ ,  $p$ -periodic with respect to both its arguments:

$$(12) \quad w(x, y) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} w(j, k) e_j^p(x) e_k^p(y).$$

For  $w(x, y) = u(x + y)$  this equality has the form

$$u(x + y) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(j + k) e_j^p(x) e_k^p(y).$$

Using this form, the composition  $u \circ (\alpha + \beta)$  can be rewritten as follows:

$$u \circ (\alpha + \beta) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(j+k) (e_j^p \circ \alpha) (e_k^p \circ \beta).$$

Passing to final values, we have, on one hand,

$$(\pi(\alpha + \beta))(u) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(j+k) \pi\alpha(e_j^p) \pi\beta(e_k^p).$$

On the other hand, since  $\phi_x \psi_y u(x)v(y) = \phi(u)\psi(v)$  for arbitrary characters  $\phi, \psi$  and functions  $u, v$  from  $\mathcal{A}$ , we have

$$\begin{aligned} (\pi\alpha \oplus \pi\beta)(u) &= (\pi\alpha)_x (\pi\beta)_y u(x+y) = \\ &= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(j+k) (\pi\alpha)_x (\pi\beta)_y e_j^p(x) e_k^p(y) = \\ &= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(j+k) \pi\alpha(e_j^p) \pi\beta(e_k^p). \end{aligned}$$

The additivity of the mapping  $\pi$  is thus proved. To prove its multiplicativity it is sufficient to verify that for any periodic function  $u$  the values of the functionals  $\pi(\alpha\beta)$  and  $\pi\alpha \odot \pi\beta$  coincide. For  $w(x, y) = u(xy)$  the equality (12) can be transformed to the form

$$u(xy) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} u(jk) e_j^p(x) e_k^p(y),$$

then the multiplicativity of the mapping  $\pi$  can be proved in the same way as its additivity.

Thus we have ascertained that  $\pi$  is a homomorphism of the ring of stabilizers  $\mathcal{S}$  into the ring of characters  $\mathcal{G}$ . Belonging to the kernel  $\ker \pi$  for a stabilizer  $\alpha$  is equivalent to the fact that  $\pi\alpha$  is the neutral element  $\theta$  of the additive group of the ring  $\mathcal{G}$ , i. e., to the fact that  $\pi\alpha(u) = u(0)$  for any function  $u$  from  $\mathcal{A}$ . Since  $\pi\alpha(u) = (u \circ \alpha)_\infty$ , then

$$\ker \pi = \{\alpha \in \mathcal{S} : \forall u \in \mathcal{A} \ (u \circ \alpha)_\infty = u(0)\} = \mathcal{C}_0,$$

where  $\mathcal{C}_0$  is the ideal of zero stabilizers.  $\square$

**PROPOSITION 45.** *The homomorphic image of  $\mathcal{S}$  is the ring  $\mathcal{G}$ .*

**PROOF.** Let  $\psi$  be an arbitrary character. As noted in Section 2.1, the algebra  $\mathcal{A}$  is the inductive limit of the increasing (with respect to inclusion) sequence of subalgebras  $\mathcal{A}_{n!}$ . As  $\mathcal{G}$  does not contain the zero functional, for almost all  $n$  the contractions  $\psi_{n!} = \psi|_{\mathcal{A}_{n!}}$  are nontrivial

and beginning from  $n = n(\psi)$ , we can define a sequence  $\alpha_n = \kappa(n!; \psi)$  which we extend as zero for smaller values of  $n$ . According to Proposition 7, we have

$$(13) \quad \alpha_{n+1} - \alpha_n = \kappa((n+1)!; \psi) - \kappa(n!; \psi) \equiv 0 \pmod{n!}.$$

Let  $u$  be an arbitrary function from  $\mathcal{A}$  and let  $p$  be its period. We set  $q$  large enough to satisfy the inequality  $N = qp \geq n(\psi)$ . It follows from (13) that for  $n \geq N$  the relation

$$\alpha_n = \alpha_N + \sum_{r=N}^{n-1} (\alpha_{r+1} - \alpha_r) \equiv \alpha_N \pmod{N!},$$

holds where, as usual, the “empty” sum is said to be zero. As  $N!$  is the period of the function  $u$ , for  $n \geq N$  we have

$$u(\alpha_n) = u(\alpha_N) = u(\kappa(N!; \psi)) = \psi_{N!}(u) = \psi(u).$$

Thus, the sequence  $\alpha$  constructed above stabilizes any function  $u$  to the final value  $(u \circ \alpha)_\infty = \psi(u)$ . In other words,  $\pi\alpha = \psi$ .  $\square$

Taking into account Proposition 44, we obtain

**COROLLARY.** *The quotient mapping  $\pi = \pi/\mathcal{C}_0$  acts as an isomorphism of the rings  $\mathcal{P}$  and  $\mathcal{G}$ .*

### 9.2. Isomorphism of the topological rings $(\mathcal{P}, \sigma)$ and $(\mathcal{G}, \gamma)$ .

The following proposition shows that  $\pi$  is not only an isomorphism of the rings  $\mathcal{P}$  and  $\mathcal{G}$ , but an homeomorphism of the topological spaces  $(\mathcal{P}, \sigma)$  and  $(\mathcal{G}, \gamma)$  as well, since here the zero neighborhoods of the ring  $(\mathcal{P}, \sigma)$  pass to the zero neighborhoods of the ring  $(\mathcal{G}, \gamma)$ .

**PROPOSITION 46.** *For each positive integer  $p$  the cluster  $V_p$  is the image of the ideal  $\mathcal{G}^{(p)}$  under the isomorphism  $\pi$ .*

**PROOF.** As stated in Proposition 42, the sequence  $\alpha$  represents an element of the ideal  $\mathcal{G}^{(p)}$  if and only if it is  $p$ -prezero stabilizer. Hence, the proposition will be proved if we show that the homomorphism  $\pi$  surjectively maps the set of  $p$ -prezero stabilizers onto the cluster  $V_p$ .

Let a stabilizer  $\alpha$  be  $p$ -prezero. Then for an arbitrary function  $u$  from  $\mathcal{A}_p$   $(\pi\alpha)(u) = (u \circ \alpha)_\infty = u(0) = \theta(u)$ . Thus  $\pi\alpha \in V_p$ .

Conversely, let us suppose that a character belongs to the cluster  $V_p$ , i. e., the functional  $\psi$  has the value  $v(0)$  for any  $p$ -periodic function  $v$ . Then the stabilizer  $\alpha$  constructed in the proof of Proposition 45 is  $p$ -prezero:  $(v \circ \alpha)_\infty = \psi(v) = v(0)$ .  $\square$

The following main result of this paper is an immediate consequence of Proposition 45 (Corollary) and Proposition 46:

**THEOREM.** *The ring  $(\mathcal{P}, \sigma)$  is algebraically and topologically isomorphic to the ring  $(\mathcal{G}, \gamma)$ .*

This means that we can identify the polyadic topological ring  $(\mathcal{P}, \sigma)$  of Prüfer, Van Dantzig and Novoselov with the topological ring of characters of the algebra  $\mathcal{A}$  consisting of all periodic functions on  $\mathbb{Z}$ .

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